Braiding statistics and symmetry-protected topological phases

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Definition of SPT phases

Gapped quantum many-body system with

- No symmetry breaking
- No fractional statistics (“short-range entangled”)
- Cannot be connected to a product state with same symmetry
- Symmetry-protected boundary modes

Basic questions about SPT phases

- **Classification**: For each symmetry group and spatial dimension, how many SPT phases are there?

  - Non-interacting fermions (Schnyder et al, Kitaev, 2008)

  - General boson systems (Chen, Gu, Liu, Wen, 2011)

- **Characterization**: How can we determine whether a microscopic model belongs to a specific SPT phase?

A simple example

- Focus on **2D** spin systems with $\mathbf{Z_2}$ (Ising) symmetry

- One non-trivial SPT phase, one trivial phase

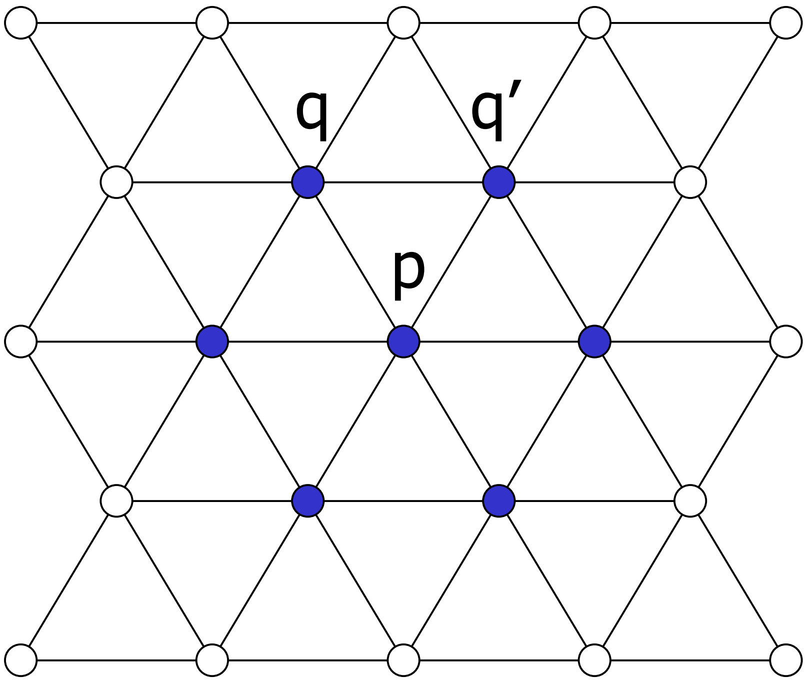
  (Chen, Gu, Liu, Wen, 2011)

$\Rightarrow$ "Two kinds of Ising paramagnets"

Two kinds of Ising paramagnets

Symmetry:

$$S = \prod_p \sigma_p^x$$



Hamiltonians:

$$H_0 = -\sum_p \sigma_p^x$$

$$H_1 = -\sum_p B_p \quad , \quad B_p = -\sigma_p^x \prod_{pqq'} i^{(1-\sigma_q^z \sigma_{q'}^z)/2}$$

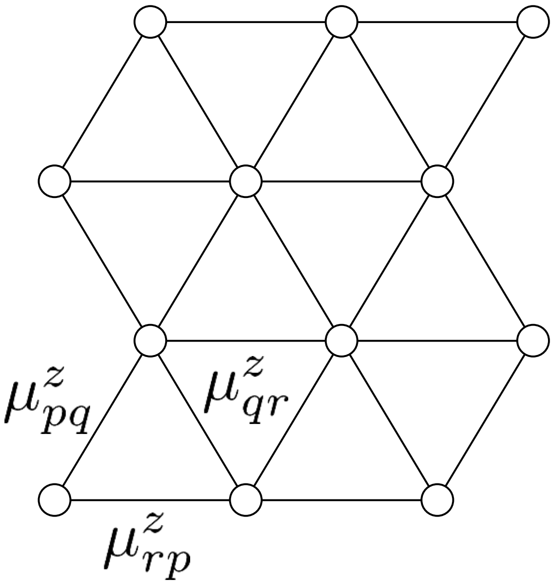
Two kinds of Ising paramagnets

1. How can we see that H_0 and H_1 belong to different phases?

2. How can we see that H_1 has a protected edge mode while H_0 does not?

Step 1: Couple to a **Z$_2$** gauge field

Z$_2$ gauge field: $\mu^z_{pq} = \pm 1$



Replace: $\sigma^z_p \sigma^z_q \quad \rightarrow \quad \sigma^z_p \mu^z_{pq} \sigma^z_q$

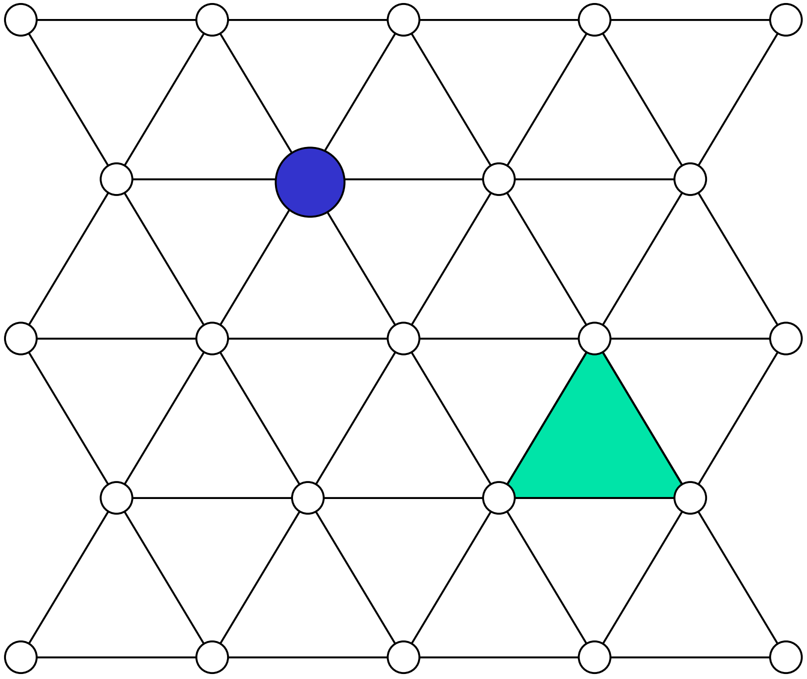
Step 2: Find basic excitations

1. "Charge": e

$$\sigma_p^x = -1 \text{ or } B_p = -1$$

2."π-vortex": m_a, m_b

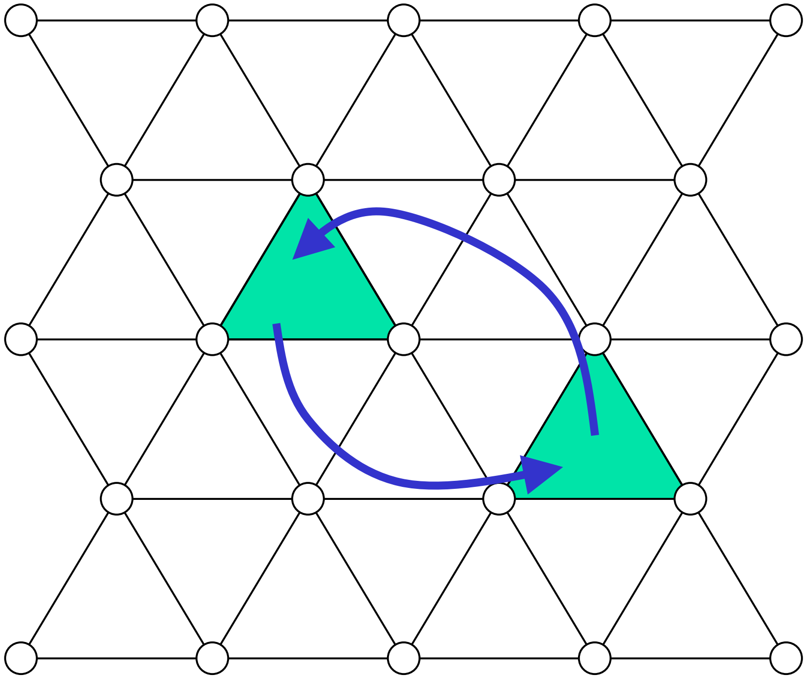
$$\mu_{pq}^z \mu_{qr}^z \mu_{rp}^z = -1$$



Step 3: Compute braiding statistics of π-vortex excitations

π–vortex:

$$\mu^z_{pq}\mu^z_{qr}\mu^z_{rp} = -1$$



$e^{i\theta} = ?$

(ML, Z. Gu, 2012)

Result for statistics

$\mathbf{H_0}$: Find $e^{i\theta} = \pm 1$

$\Rightarrow$ π-vortices are bosons or fermions

$\mathbf{H_1}$: Find $e^{i\theta} = \pm i$

$\Rightarrow$ π-vortices are semions or anti-semions

(ML, Z. Gu, 2012)

Braiding statistics approach

- Proves two models H_0, H_1 belong to distinct phases
- Provides proof that H_1 has protected edge modes
- Analogous results for any group cohomology model with unitary, abelian symmetry group

Repeat program in 3D

1. Take short range entangled spin model with symmetry group G

2. Gauge the symmetry

3. Study braiding statistics of excitations in resulting gauge theory

4. Focus on simple case: $G = (\mathbf{Z_N})^K$

Excitations in $(\mathbf{Z_N})^K$ gauge theories

1. "Charges"

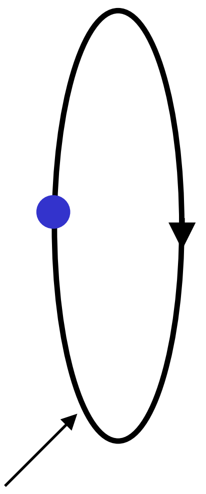
- Characterized by gauge charge:

$$q = (q_1,\ldots,q_K) \quad , \quad q_i = \text{integer (mod N)}$$

2. "Vortex loops"

- Characterized by gauge flux:

$$\phi = (\phi_1,\ldots,\phi_K), \quad \phi_i = (2\pi/N) \cdot \text{integer}$$

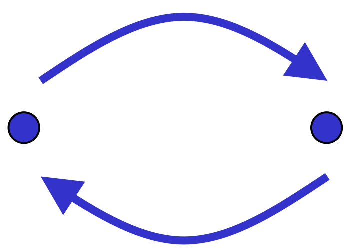


flux ϕ

- Vortex loops can also carry gauge charge

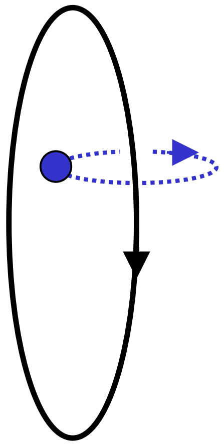
Braiding statistics in $(\mathbf{Z_N})^K$ gauge theories

Charge-charge



$$\theta = 0$$

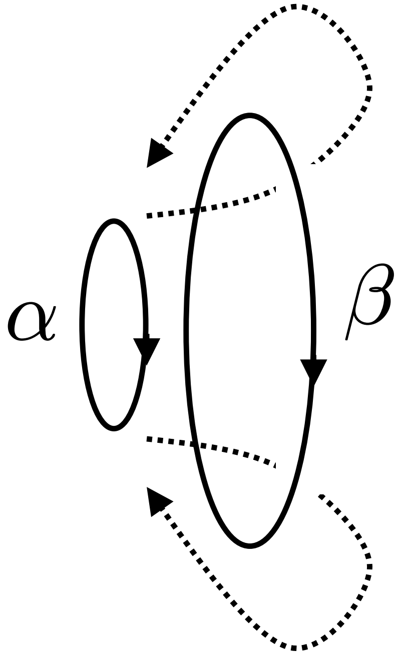
Charge-loop



$$\theta = q \cdot \phi$$

Braiding statistics in $(\mathbf{Z_N})^K$ gauge theories

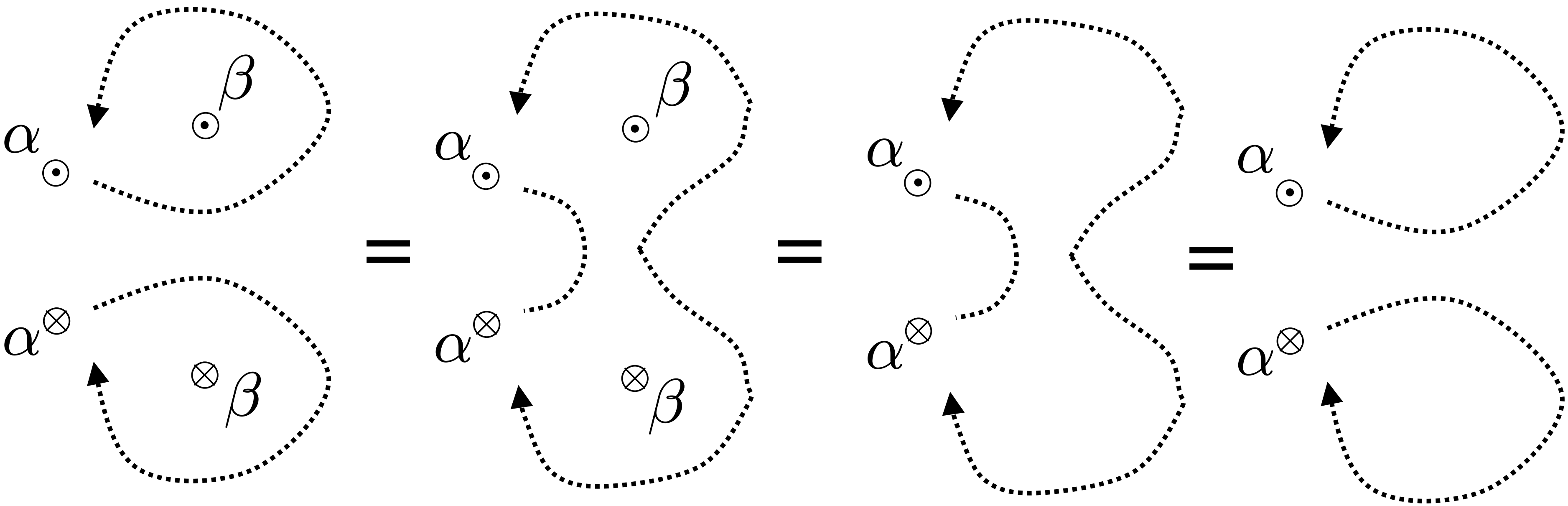
Loop-loop



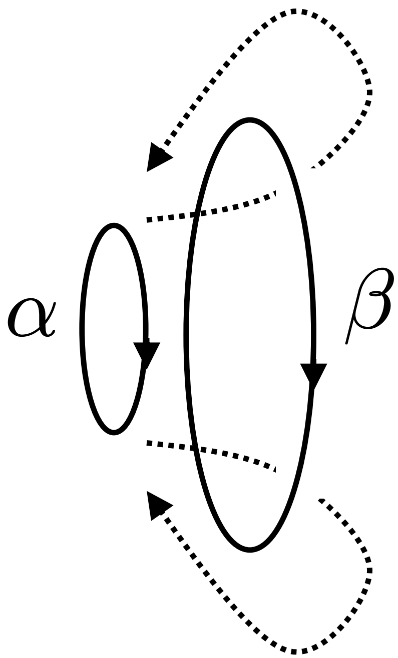
$\theta_{\alpha\beta}$ = ?

Braiding statistics in $(\mathbf{Z_N})^K$ gauge theories

If α, β are neutral, $\theta_{\alpha\beta} = 0$:



Braiding statistics in $(\mathbf{Z_N})^K$ gauge theories



General case:

$$\theta_{\alpha\beta} = q_\alpha \cdot \phi_\beta + q_\beta \cdot \phi_\alpha$$

q_α = charge carried by α
ϕ_α = flux carried by α

Braiding statistics in $(\mathbf{Z_N})^K$ gauge theories

Charge-charge: $\theta = 0$

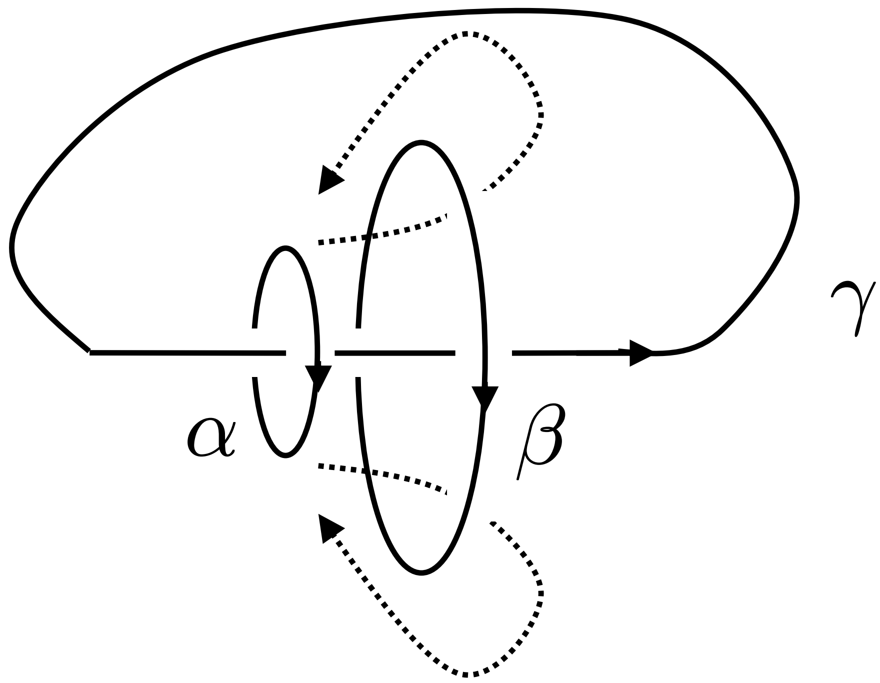
Charge-loop: $\theta = q \cdot \phi$

Loop-loop: $\theta_{\alpha\beta} = q_\alpha \cdot \phi_\beta + q_\beta \cdot \phi_\alpha$

Independent of properties of bosonic matter!

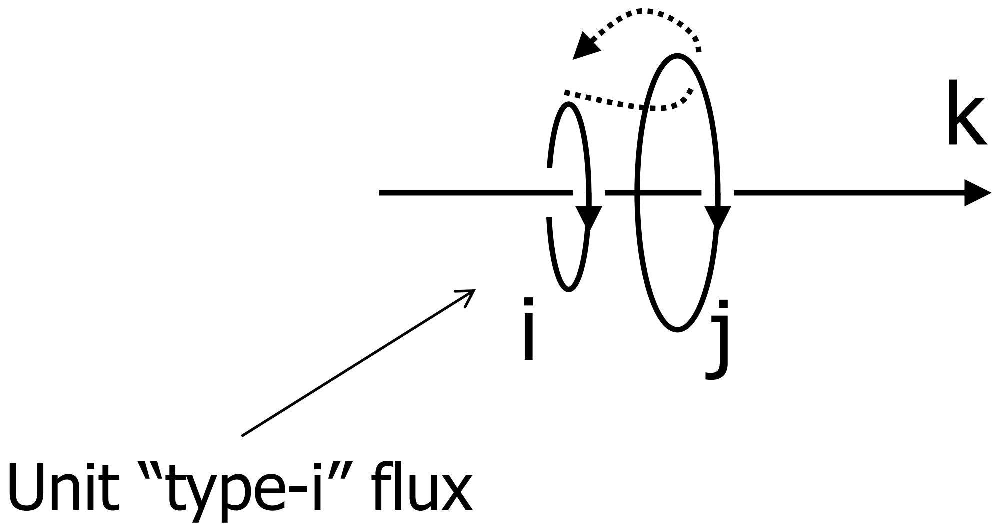
(C. Wang, ML, 2014)
(Jiang, Mesaros, Ran, 2014)

Three-loop braiding statistics



$\theta_{\alpha\beta,c}$ where ϕ_γ = 2πc/N

Minimal data for 3-loop statistics



Define:

$$\Theta_{ij,k} = N \cdot \theta(\text{above process}) \pmod{2\pi}$$

Example: $\mathbf{Z_N \times Z_N}$

N^2 different exactly solvable group cohomology models labeled by (p_1, p_2). Statistics:

$$
\begin{aligned}
\Theta_{11,1} &= 0 \\
\Theta_{12,1} &= 2\pi p_1/N \\
\Theta_{22,1} &= -4\pi p_2/N \\
\Theta_{11,2} &= -4\pi p_1/N \\
\Theta_{12,2} &= 2\pi p_2/N \\
\Theta_{22,2} &= 0
\end{aligned}
$$

3-loop statistics distinguishes different group cohomology models

Summary

- Braiding statistics distinguishes many 2D/3D SPT phases

- 3D case requires “three-loop” statistics $\theta_{\alpha\beta,c}$